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1987 J. Phys. A: Math. Gen. 20 3653

(http://iopscience.iop.org/0305-4470/20/12/019)

J. Phys. A: Math. Gen. 20 (1987) 3653-3666. Printed in the UK

The modulation of weakly non-linear ion acoustic plasma waves near the marginal state of instability

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Received 8 December 1986, in final form 24 February 1987

Abstract. The modulation of a one-dimensional weakly non-linear quasimonochromatic ion acoustic plasma wave (the carrier) is considered. It is well known that the carrier is modulationally unstable for wavenumbers k larger than a critical wavenumber k_c, and that when k is not near k_c the modulations of the carrier are governed by a non-linear Schrödinger (NS) equation. We show that when k is near k_c, and under certain assumptions, the modulations are governed by a modified form of the NS equation that involves higher-order non-linearities, and that the correct critical wavenumber for marginal modulational instability is slightly different from k_c.

1. Introduction

Suppose the slow amplitude modulation of a one-dimensional weakly non-linear quasimonochromatic purely dispersive wave (the carrier wave) is characterised by a small parameter ε. It is well known that if the non-linearity is assumed to be of $O(\varepsilon)$ then the evolution of the complex amplitude φ of the carrier is governed by the non-linear Schrödinger (NS) equation

$$\mathrm{i}\frac{\partial\varphi}{\partial\tau_2}+p\frac{\partial^2\varphi}{\partial\xi_1^2}=q|\varphi|^2\varphi \tag{1.1}$$

where $\tau_2=\varepsilon^2 t$, $\xi_1=\varepsilon(x-V_g t)$, t and x are time and space coordinates respectively, V_g and k are respectively the group velocity and wavenumber of the carrier wave, and $p=\frac{1}{2}\mathrm{d}V_g/\mathrm{d}k$ and q are real functions of k. Jeffrey and Kawahara (1982) give a representative selection of references. Furthermore the criterion for the modulational instability of the carrier is

$$pq<0. \tag{1.2}$$

For many physical systems pq has just one real zero, at some critical wavenumber k_c, and $pq\sim k-k_c$ for k near k_c. The criterion (1.2) may then be written $k<k_c$ or $k>k_c$ depending on whether pq is increasing or decreasing at $k=k_c$. It appears then that the carrier is marginally modulationally unstable at $k=k_c$.

Kakutani and Michihiro (1983) point out that (1.1) and (1.2) are not appropriate near the marginal state. Their argument has been discussed in detail in a previous paper (Parkes 1987, hereafter referred to as I). Essentially they argue that near the marginal state a different ordering should be used to intensify the effect of the non-linearity; this leads to a new governing equation for φ to replace (1.1), and to a revised modulational instability criterion to replace (1.2). As an illustration they

0305-4470/87/123653+14$02.50

considered the modulation of Stokes waves (i.e. gravity waves on water of uniform depth) near the marginal state. By assuming that the effect of the non-linearity is then of $O(\varepsilon^{1/2})$ instead of $O(\varepsilon)$ they derived a governing equation for φ of the form

$$\mathrm{i}\frac{\partial\varphi}{\partial\tau_2}+p\frac{\partial^2\varphi}{\partial\xi_1^2}=q_1|\varphi|^2\varphi+q_2|\varphi|^4\varphi+\mathrm{i}q_3\varphi\frac{\partial}{\partial\xi_1}|\varphi|^2+\mathrm{i}q_4|\varphi|^2\frac{\partial\varphi}{\partial\xi_1} \tag{1.3}$$

where $q_1(=q/\varepsilon)$, q_2, q_3 and q_4 are O(1) real functions of k, which is assumed to be such that $k-k_c$ is of $O(\varepsilon)$. We shall refer to (1.3) as the modified non-linear Schrödinger (MNS) equation. Kakutani and Michihiro also derived a revised modulational instability criterion associated with (1.3). In the notation of I this criterion may be written

$$pq<\varepsilon r \tag{1.4}$$

where

$$r=\kappa pq_4-(q_3^2+4pq_2)|\varphi_0|^2/2.$$

Here κ is a measure of the spread of wavenumbers about the dominant wavenumber in the carrier and $|\varphi_0|$ is, to lowest order, the the amplitude of the carrier. The criterion (1.4) effectively gives a revised value for the critical wavenumber.

The modulation of Stokes waves has also been considered by Johnson (1977). He obtained an equation slightly different from (1.3) and a corresponding modulational instability criterion different from (1.4).

In I we considered an arbitrary non-linear purely dispersive system in which a single dependent variable u satisfies an equation of the form $\mathscr{L}u=N$, where $\mathscr{L}$ is a linear operator involving the differential operators $\partial/\partial t$ and $\partial/\partial x$, and N represents all the non-linear terms. Assuming the non-linearity is of $O(\varepsilon^{1/2})$ near the marginal state we formally derived (1.3). However, as we remarked in I, many purely dispersive physical systems are described by the more general class of quasilinear partial differential equations

$$A(U)\frac{\partial U}{\partial t}+B(U)\frac{\partial U}{\partial x}+C(U)=0 \tag{1.5}$$

where $U=(u_i)$ is an n-component column vector, and the $n\times n$ matrices A, B and the n-component column vector $C=(c_i)$ are functions of u_i, all the quantities being real. The purpose of this paper is to discuss a particular example of (1.5).

A simple two-fluid model of a plasma, suitable for investigating the propagation of ion acoustic waves, has been studied by many authors. In particular Shimizu and Ichikawa (1972)†, using the reductive perturbation method, obtained a NS equation for the complex amplitude of the perturbation to the ion density, and Kakutani and Sugimoto (1974), using the Krylov-Bogoliubov-Mitropolsky perturbation method, obtained a NS equation for the complex amplitude of the electric field. Implicit in both these derivations was the assumption that k is not near k_c. Kakutani and Sugimoto also evaluated k_c and showed that the modulational instability criterion is $k>k_c$. In the present paper we consider the propagation of ion acoustic waves when k is near k_c. Using the derivative expansion perturbation method (Kawahara 1973) we derive, under certain restrictions, the MNS equation that replaces the NS equation of Kakutani and Sugimoto and obtain a revised modulational instability criterion.

† There are some minor errors in Shimizu and Ichikawa (1972) that are corrected in Ichikawa and Watanabe (1977).

In § 2 we obtain the basic equations in the form (1.5), and in § 3 we show how to apply the derivative expansion procedure to them. In § 4 we quickly recover the NS equation and modulational instability criterion as obtained in a different way by Kakutani and Sugimoto (1974), the derivation being on the assumption that k is not near k_c. In § 5 we consider the marginal state and derive the MNS equation. In § 6 we derive the corresponding revised modulational instability criterion.

2. The basic equations

We consider one-dimensional ion acoustic waves propagating in a magnetic-field-free collisionless plasma consisting of cold ions and isothermal electrons. Assuming a two-fluid model and neglecting the effects of Landau damping and electron inertia, the governing equations may be written in non-dimensional form as

$$\begin{aligned} &\partial v/\partial t + v\partial v/\partial x = E \\ &\partial n/\partial t + \partial (nv)/\partial x = 0 \\ &\partial n_e/\partial x = -n_e E \\ &\partial E/\partial x = n - n_e \end{aligned} \tag{2.1}$$

where n, n_e, v, E are respectively the non-dimensional ion density, electron density, ion fluid velocity and electric field. The reference density, speed and time are respectively the undisturbed density n_0, the ion sound speed $c_s = (\mathcal{K}T_e/m_i)^{1/2}$ and $(\omega_{pi})^{-1}$, where $\omega_{pi} = (n_0 e^2/\varepsilon_0 m_i)^{1/2}$ is the ion plasma frequency. Here $\mathcal{K}$ is Boltzmann's constant, T_e the constant electron temperature, m_i the ion mass, e the electron charge and ε_0 the vacuum permittivity. The reference electric field is $m_i\omega_{pi}c_s/e$. Equations (2.1) are the basic equations used in Shimizu and Ichikawa (1972) and Kakutani and Sugimoto (1974).

Equations (2.1) can be cast into the form (1.5) in many ways but we find it helpful to have A and B symmetric. Hence we take

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & v & 0 & 0 \\ v & n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$C = [-E, 0, n_e E, n_e - n]^{\mathrm{T}} \qquad U = [n, v, n_e, E]^{\mathrm{T}}$$

where T denotes the transpose. The unperturbed system corresponds to the constant solution $U^{(0)} = [1, 0, 1, 0]^{\mathrm{T}}$ for which $C(U^{(0)}) = 0$. Using this it proves convenient to rewrite (1.5) in the form

$$LU = M \tag{2.2}$$

where

$$L = A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x} + \nabla C_0 \tag{2.3}$$

$$M = -C + (\nabla C_0)U = [0, 0, (1 - n_e)E, 0]^{\mathrm{T}}. \tag{2.4}$$

∇C_0 is defined by $(\nabla C_0)_{ij} = \partial c_i/\partial u_j$ evaluated at $U = U^{(0)}$ so that

$$\nabla C_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

3. The derivative expansion method

In this section we show how to apply the derivative expansion procedure (Kawahara 1973) to the system of equations (in the form (1.5)) that govern ion acoustic waves and establish the general method of solution. Details of the calculation for the non-marginal and marginal states of modulational instability are given in §§ 4 and 5, respectively.

We introduce the extended set of independent variables

$$t_0 = t \qquad x_0 = x \qquad \tau_i = \varepsilon^i t \qquad \xi_i = \varepsilon^i (x - V_g t) \qquad (i = 1, 2, \ldots, N)$$

where ε is a small parameter characterising the slow modulation. As in I, it is sufficient to take $N = 2$ here. Thus defined t_0, x_0 are the variables appropriate to the 'fast' oscillations of the carrier, and τ_1, ξ_1, τ_2, ξ_2 are 'slow' variables appropriate to the slow modulations in a reference frame moving with the velocity V_g. The time and space derivatives in (2.3) may now be expressed as the derivative expansions

$$\begin{aligned} \frac{\partial}{\partial t} &\equiv -\omega \frac{\partial}{\partial \vartheta} + \varepsilon \left(\frac{\partial}{\partial \tau_1} - V_g \frac{\partial}{\partial \xi_1} \right) + \varepsilon^2 \left(\frac{\partial}{\partial \tau_2} - V_g \frac{\partial}{\partial \xi_2} \right) \\ \frac{\partial}{\partial x} &\equiv k \frac{\partial}{\partial \vartheta} + \varepsilon \frac{\partial}{\partial \xi_1} + \varepsilon^2 \frac{\partial}{\partial \xi_2} \end{aligned} \qquad (3.1)$$

where $\vartheta = kx_0 - \omega t_0$, ω and k are, to lowest order, the phase of the fast oscillations and the non-dimensional frequency and wavenumber.

First we consider the non-marginal state. The non-linearity is assumed to be of $O(\varepsilon)$, as in Shimizu and Ichikawa (1972) and Kakutani and Sugimoto (1974), so that U may be written

$$U = U^{(0)} + \sum_{m=1}^{3} \varepsilon^m U^{(m)}(\vartheta, \tau_1, \xi_1, \tau_2, \xi_2) + O(\varepsilon^4) \qquad (3.2)$$

where, for $m \geq 1$,

$$U^{(m)} = [n^{(m)}, v^{(m)}, n_e^{(m)}, E^{(m)}]^T \qquad (3.3)$$

with $U^{(m)} = 0$ in the unperturbed state where there is no wave. Using (3.2) we may express B as

$$B = B_0 + \sum_{m=1}^{2} \varepsilon^m B_m(\vartheta, \tau_1, \xi_1, \tau_2, \xi_2) + O(\varepsilon^3) \qquad (3.4)$$

where

$$B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B_m = \begin{bmatrix} 0 & v^{(m)} & 0 & 0 \\ v^{(m)} & n^{(m)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (3.5)$$

Substitution of (3.1), (3.2) and (3.4) into (2.3) and (2.4) gives the expansions

$$L=\sum_{m=0}^{2}\varepsilon^{m}L_m+\mathrm{O}(\varepsilon^{3}) \tag{3.6}$$

$$M=\sum_{m=2}^{3}\varepsilon^{m}M_m(\vartheta,\tau_1,\xi_1,\tau_2,\xi_2)+\mathrm{O}(\varepsilon^{4}) \tag{3.7}$$

where the L_m are given in appendix 1 and

$$M_m=\left[0,0,-\sum_{j=1}^{m}n_e^{(j)}E^{(m-j)},0\right]^{\mathrm{T}}. \tag{3.8}$$

Substituting (3.2), (3.6) and (3.7) into (2.2) and equating like powers of ε, we obtain the hierarchy of equations

$$\mathrm{O}(\varepsilon^{m}):\ L_0U^{(m)}=\begin{cases}0 & m=1\\ M_m-\sum_{j=1}^{m-1}L_jU^{(m-j)} & m=2,3.\end{cases} \tag{3.9}$$

In order to investigate the behaviour of the slow modulations near marginal instability we intensify the non-linear effects by assuming the non-linearity is of $\mathrm{O}(\varepsilon^{1/2})$ as in Kakutani and Michihiro (1983). We write U as

$$U=U^{(0)}+\sum_{m=1}^{6}\varepsilon^{m/2}U^{(m)}(\vartheta,\tau_1,\xi_2,\tau_2,\xi_2)+\mathrm{O}(\varepsilon^{7/2}) \tag{3.10}$$

where the $U^{(m)}$ are defined by (3.3), and then

$$B=B_0+\sum_{m=1}^{5}\varepsilon^{m/2}U^{(m)}(\vartheta,\tau_1,\xi_1,\tau_2,\xi_2)+\mathrm{O}(\varepsilon^{7/2}) \tag{3.11}$$

where B_0 and B_m are as given by (3.5). Substitution of (3.1), (3.10) and (3.11) into (2.3) and (2.4) gives the expansions

$$L=\sum_{m=0}^{5}\varepsilon^{m/2}L_m+\mathrm{O}(\varepsilon^{3}) \tag{3.12}$$

$$M=\sum_{m=2}^{6}\varepsilon^{m/2}M_m(\vartheta,\tau_1,\xi_1,\tau_2,\xi_2)+\mathrm{O}(\varepsilon^{7/2}) \tag{3.13}$$

where the L_m are given in appendix 1 and the M_m are given by (3.8). Substituting (3.2), (3.12) and (3.13) into (2.2) and equating like powers of ε, we obtain the hierarchy of equations

$$\mathrm{O}(\varepsilon^{m/2}):\ L_0U^{(m)}=\begin{cases}0 & m=1\\ M_m-\sum_{j=1}^{m-1}L_jU^{(m-j)} & m=2,\ldots,6.\end{cases} \tag{3.14}$$

We observe that the equations in (3.9) and (3.14) corresponding to $m=1$ are the same. We assume that the solution to these equations is the quasimonochromatic wave

$$U^{(1)}=\varphi(\tau_1,\xi_1,\tau_2,\xi_2)K\exp(\mathrm{i}\vartheta)+\mathrm{cc} \tag{3.15}$$

where φ is a complex scalar function and K is a constant column vector. Here, and subsequently, cc is used to denote the complex conjugate of all the preceding terms. Substituting (3.15) into the $m=1$ equation in (3.9) or (3.14) we deduce that there is a non-trivial solution for K provided ω and k satisfy

$$\mathscr{D}(\omega, k) \equiv \det\{D_0(\omega, k)\} = 0$$

where

$$D_0(\omega, k) = -\mathrm{i}\omega A + \mathrm{i}kB_0 + \nabla C_0 = \begin{bmatrix} 0 & -\mathrm{i}\omega & 0 & -1 \\ -\mathrm{i}\omega & \mathrm{i}k & 0 & 0 \\ 0 & 0 & \mathrm{i}k & 1 \\ -1 & 0 & 1 & \mathrm{i}k \end{bmatrix}$$

so that

$$\mathscr{D}(\omega, k) \equiv -\omega^2(k^2+1) + k^2 = 0. \tag{3.16}$$

Equation (3.16) is the linear dispersion relation for ion acoustic waves from which we obtain

$$V_g = \mathrm{d}\omega/\mathrm{d}k = \omega^3/k^3 \qquad p = \tfrac{1}{2}\,\mathrm{d}V_g/\mathrm{d}k = -3\omega^5/2k^4.$$

We note in passing that (3.16) has been used in various places in the rest of this paper to effect some simplifications. We observe that if $k \neq 0$ then $\mathscr{D}(n\omega, nk) \neq 0$ for $n = 2, 3, \ldots$ so that the inverse of the matrix $D_0(n\omega, nk)$ exists for these values of n. Also, as rank $[D_0(\omega, k)] = 3$ and K must satisfy $D_0 K = 0$, one component of K is arbitrary. Hence we may choose the fourth component to be unity so that φ is the amplitude of the electric field E. Then (3.15) becomes

$$U^{(1)} = [\mathrm{i}k/\omega^2, \mathrm{i}/\omega, \mathrm{i}/k, 1]^{\mathrm{T}} \varphi \exp(\mathrm{i}\vartheta) + \mathrm{cc}. \tag{3.17}$$

Explicit solutions to (3.9) and (3.14) for $m > 1$ are given in §§ 4 and 5 respectively. Here we summarise the method of solution and obtain non-secular conditions. We find that, for each $m > 1$, equation (3.9) or (3.14) may be written

$$L_0 U^{(m)} = \tilde{U}_0^{(m)} + \left(\sum_{n=1}^{m} \tilde{U}_n^{(m)} \exp(\mathrm{i}n\vartheta) + \mathrm{cc} \right) \tag{3.18}$$

where the $\tilde{U}_n^{(m)}$ $(n = 0, 1, \ldots, m)$ are independent of ϑ and are determined by the solutions $U^{(j)}$ $(j = 1, \ldots, m-1)$ to previous equations in the hierarchy. As we require solutions involving no secular terms we assume a solution to (3.18) of the form

$$U^{(m)} = U_0^{(m)} + \left(\sum_{n=1}^{m} U_n^{(m)} \exp(\mathrm{i}n\vartheta) + \mathrm{cc} \right)$$

where the $U_n^{(m)}$ $(n = 0, 1, \ldots, m)$ are independent of φ. This assumption imposes up to two conditions at each order, namely (3.21) and (3.24) below, that may be regarded as 'non-secular conditions'. The $U_n^{(m)}$ are determined as follows.

The function $U_0^{(m)}$ satisfies $L_0 U_0^{(m)} = \tilde{U}_0^{(m)}$ from which we obtain

$$(\nabla C_0) U_0^{(m)} = \tilde{U}_0^{(m)}. \tag{3.19}$$

As $\mathrm{rank}(\nabla C_0) = 2$, (3.19) has a solution provided that

$$\mathrm{rank}(\nabla C_0, \tilde{U}_0^{(m)}) = 2 \tag{3.20}$$

where $(\nabla C_0, \tilde{U}_0^{(m)})$ is the 4×5 augmented matrix whose fifth column consists of the components of $\tilde{U}_0^{(m)}$. If we write $\tilde{U}_0^{(m)} = [r_1, r_2, r_3, r_4]^{\mathrm{T}}$, the condition (3.20) implies that

$$r_1 + r_3 = 0 \qquad r_2 = 0 \tag{3.21}$$

and then the solution to (3.19) is $U_0^{(m)} = [\lambda_m, \mu_m, \lambda_m + r_4, r_3]^{\mathrm{T}}$, where λ_m, μ_m are arbitrary real functions of the slow variables.

The function $U_1^{(m)}$ satisfies $L_0 U_1^{(m)} \exp(\mathrm{i}\vartheta) = \tilde{U}_1^{(m)} \exp(\mathrm{i}\vartheta)$ from which we obtain

$$D_0(\omega, k) U_1^{(m)} = \tilde{U}_1^{(m)}. \tag{3.22}$$

As rank $(D_0) = 3$, (3.22) has a solution provided that

$$\operatorname{rank}(D_0, \tilde{U}_1^{(m)}) = 3. \tag{3.22}$$

If we write $\tilde{U}_1^{(m)} = [s_1, s_2, s_3, s_4]^{\mathrm{T}}$, the condition (3.23) implies that

$$-k^2 s_1 - \omega k s_2 - \omega^2 s_3 + \mathrm{i}\omega^2 k s_4 = 0 \tag{3.24}$$

and then the solution to (3.22) is

$$U_1^{(m)} = \mathrm{i}[(ks_1 + \omega s_2)/\omega^2, s_1/\omega, -s_3/k, 0]^{\mathrm{T}} + h_m[\mathrm{i}k/\omega^2, \mathrm{i}/\omega, \mathrm{i}/k, 1]^{\mathrm{T}} \tag{3.25}$$

where h_m is an arbitrary function of the slow variables. The second term in (3.25) is the solution to the homogeneous version of (3.22). We may ignore it as it may be absorbed into (3.15) by suitably redefining φ.

The $U_n^{(m)}$ $(n = 2, \ldots, m)$ satisfy $L_0 U_n^{(m)} \exp(\mathrm{i}n\vartheta) = \tilde{U}_n^m \exp(\mathrm{i}n\vartheta)$ from which we obtain $D_0(n\omega, nk) U_n^{(m)} = \tilde{U}_n^{(m)}$. The solution is simply $U_n^{(m)} = D^{-1}(n\omega, nk)\tilde{U}_n^{(m)}$.

In §§ 4 and 5 we summarise the explicit results obtained at each order for $m > 1$ by applying the above methodology to (3.9) and (3.14), respectively, and investigate the consequences of the non-secular conditions. The vector coefficients that are not stated explicitly in §§ 4 and 5 may be found in appendix 2.

4. Derivation of the non-linear Schrödinger equation

In this section we are concerned exclusively with the hierarchy (3.9). We recover the results of Kakutani and Sugimoto (1974) that are appropriate when k is not near k_c.

At $O(\varepsilon^2)$ there is one non-secular condition, namely (3.24), which gives

$$\partial\varphi/\partial\tau_1 = 0 \tag{4.1}$$

and then the solution to (3.9) is

$$U_0^{(2)} = [\lambda_2, \mu_2, \lambda_2, 0]^{\mathrm{T}} \qquad U_1^{(2)} = \alpha^{(2)}\partial\varphi/\partial\xi_1 \qquad U_2^{(2)} = \beta^{(2)}\varphi^2.$$

We assume that $U_0^{(2)}$ depends on τ_1 and ξ_1 through φ and φ^* only, where * denotes the complex conjugate, and hence, in view of (4.1), that it is independent of τ_1.

At $O(\varepsilon^3)$ there are two non-secular conditions. The condition (3.21) gives

$$\begin{aligned} \frac{\partial\mu_2}{\partial\tau_1} - V_g\frac{\partial\mu_2}{\partial\xi_1} + \frac{\partial\lambda_2}{\partial\xi_1} + \frac{\partial}{\partial\xi_1}|\varphi|^2 = 0 \\ \frac{\partial\lambda_2}{\partial\tau_1} - V_g\frac{\partial\lambda_2}{\partial\xi_1} + \frac{\partial\mu_2}{\partial\xi_1} + \frac{2k}{\omega^3}\frac{\partial}{\partial\xi_1}|\varphi|^2 = 0 \end{aligned} \tag{4.2}$$

which may be solved to give

$$\lambda_2 = \hat{\lambda}_2|\varphi|^2 + \rho_2 \qquad \mu_2 = \hat{\mu}_2|\varphi|^2 + \sigma_2 \tag{4.3}$$

where

$$\hat{\lambda}_2 = -\frac{k^2(k^2+2)}{\omega^6(k^4+3k^2+3)} \qquad \hat{\mu}_2 = -\frac{(2k^6+6k^4+7k^2+2)}{\omega^3 k(k^4+3k^2+3)}$$

and ρ_2, σ_2 are arbitrary real functions of τ_2 and ξ_2. If we assume that, for $m>1$, $U^{(m)}=0$ when there is no wave, i.e. when $\varphi=0$, then we may set $\rho_2=0$ and $\sigma_2=0$. The other non-secular condition (3.24) now gives the NS equation (1.1) with

$$q(k) = \frac{\omega^3(3k^{10}+6k^8-6k^6-29k^4-30k^2-12)}{12k^6(k^4+3k^2+3)} \tag{4.4}$$

where we have assumed that q is of O(1).

The (real) critical wavenumber k_c—for which $q(k_c)=0$—satisfies

$$3k_c^{10}+6k_c^8-6k_c^6-29k_c^4-30k_c^2-12=0 \tag{4.5}$$

from which we find that $k_c=1.471$ (to three decimal places). The modulational instability criterion (1.2) may then be written

$$k > k_c. \tag{4.6}$$

5. Derivation of the modified non-linear Schrödinger equation

In this section we investigate the hierarchy (3.14) in order to show how the results of the previous section are modified when k is near to k_c.

At O(ε) there are no non-secular conditions and the solution to (3.14) is

$$U_0^{(2)} = [\lambda_2, \mu_2, \lambda_2, 0]^{\mathrm{T}} \qquad U_1^{(2)} = 0 \qquad U_2^{(2)} = \beta^{(2)}\varphi^2.$$

At O($\varepsilon^{3/2}$) there is one non-secular condition, namely (3.24), which gives

$$\mathrm{i}\frac{\partial\varphi}{\partial\tau_1} = \left(\frac{\omega^3\lambda_2}{2} + k\mu_2\right)\varphi + \frac{\omega^3}{12k^6}(24k^8+81k^6+93k^4+42k^2+4)|\varphi|^2\varphi. \tag{5.1}$$

The solution to (3.14) is

$$U_0^{(3)} = [\lambda_3, \mu_3, \lambda_3, 0]^{\mathrm{T}} \qquad U_2^{(3)} = 0 \qquad U_3^{(3)} = \delta^{(3)}\varphi^3$$

$$U_1^{(3)} = \alpha^{(3)}\frac{\partial\varphi}{\partial\tau_1} + \beta^{(3)}\frac{\partial\varphi}{\partial\xi_1} + \bar{\gamma}^{(3)}|\varphi|^2\varphi + \mathrm{i}\left[\frac{2k^2\mu_2}{\omega^3} + \frac{k\lambda_2}{\omega^2}, \frac{k\mu_2}{\omega^2}, \frac{\lambda_2}{k}, 0\right]^{\mathrm{T}}\varphi. \tag{5.2}$$

At O(ε^2) the non-secular condition (3.21) is just (4.2). This may be integrated to give the expressions (4.3) for λ_2 and μ_2, where we have used the fact that $(\partial/\partial\tau_1)|\varphi|^2=0$, as may be shown by combining (5.1) with its complex conjugate. As before we set $\rho_2=0$ and $\sigma_2=0$.

Returning to the O($\varepsilon^{3/2}$) problem, insertion of (4.3) into (5.1) apparently gives

$$\mathrm{i}\frac{\partial\varphi}{\partial\tau_1} = q|\varphi|^2\varphi \tag{5.3}$$

where q is given by (4.4). In § 4 we assumed that k was not near k_c and that q was of O(1). Here, however, we are considering the marginal state and we assume that $\Delta k = k - k_c$ is of $O(\varepsilon)$ and write $q = \varepsilon q_1$, where q_1 is of O(1) and is given approximately by

$$q_1 = \frac{\Delta k}{\varepsilon}\left(\frac{\mathrm{d}q}{\mathrm{d}k}\right)_{k=k_c} = \frac{\Delta k}{\varepsilon}\left(\frac{\omega^3(15k^8+24k^6-18k^4-58k^2-30)}{6k^5(k^4+3k^2+3)}\right)_{k=k_c}. \tag{5.4}$$

We have used (4.5) in obtaining (5.4). Hence at $O(\varepsilon^{3/2})$ equation (5.3) becomes

$$\partial\varphi/\partial\tau_1 = 0 \tag{5.5}$$

and the right-hand side of (5.3) is shifted to the corresponding non-secular condition at $O(\varepsilon^{5/2})$. Also insertion of (4.3) and (5.5) into (5.2) revises the expression for $U_1^{(3)}$ to

$$U_1^{(3)} = \beta^{(3)}\frac{\partial\varphi}{\partial\xi_1} + \gamma^{(3)}|\varphi|^2\varphi.$$

Continuing now with the $O(\varepsilon^2)$ problem, and hereafter incorporating (4.3) and (5.5) into our results, the solution to (3.14) is

$$U_0^{(4)} = \left[\lambda_4, \mu_4, \lambda_4, \frac{k^8+6k^2+12k^4+10k^2+2}{k^4(k^4+3k^2+3)}\frac{\partial}{\partial\xi_1}|\varphi|^2\right]^{\mathrm{T}}$$

$$U_2^{(4)} = \alpha^{(4)}\varphi\,\partial\varphi/\partial\xi_1 + \beta^{(4)}|\varphi|^2\varphi^2 \qquad U_3^{(4)} = 0.$$

As $U_1^{(4)}$ and $U_4^{(4)}$ play no part in the subsequent calculations, expressions for them are not given here.

At $O(\varepsilon^{5/2})$ we have

$$\begin{aligned}\tilde{U}_1^{(5)} = {} & \tilde{\alpha}^{(5)}\frac{\partial\varphi}{\partial\tau_2} + \tilde{\beta}^{(5)}\frac{\partial\varphi}{\partial\xi_2} + \tilde{\gamma}^{(5)}\frac{\partial^2\varphi}{\partial\xi_1^2} + \tilde{\delta}^{(5)}\varphi\frac{\partial}{\partial\xi_1}|\varphi|^2 + \tilde{\varepsilon}^{(5)}|\varphi|^2\frac{\partial\varphi}{\partial\xi_1} \\ & + \tilde{\zeta}^{(5)}|\varphi|^4\varphi + \left[\frac{k\mu_4}{\omega}, \frac{k\lambda_4}{\omega} + \frac{k^2\mu_4}{\omega^2}, -\lambda_4, 0\right]^{\mathrm{T}}\varphi\end{aligned} \tag{5.6}$$

from which the non-secular condition (3.24) gives

$$\mathrm{i}\frac{\partial\varphi}{\partial\tau_2} + p\frac{\partial^2\varphi}{\partial\xi_1^2} = \left(\frac{\omega^3\lambda_4}{2} + k\mu_4\right)\varphi + q_1|\varphi|^2\varphi + m_2|\varphi|^4\varphi + \mathrm{i}m_3\varphi\frac{\partial}{\partial\xi_1}|\varphi|^2 + \mathrm{i}m_4|\varphi|^2\frac{\partial\varphi}{\partial\xi_1}. \tag{5.7}$$

As explained earlier, the second term on the right-hand side of (5.7) has been shifted from the corresponding condition at $O(\varepsilon^{3/2})$, namely (5.3). The coefficients m_2, m_3, m_4 are given in appendix 3.

We note in passing that from the non-secular conditions (3.24) and (3.21), at $O(\varepsilon^2)$ and $O(\varepsilon^{5/2})$, respectively, we may deduce that $\lambda_3 = 0$, $\mu_3 = 0$. However this information is not required in the present calculation.

Equation (5.7) is almost the desired MNS equation. It only remains for us to determine λ_4 and μ_4. To do this we need to go to the next order, and there we find that the only information we need about the solution at $O(\varepsilon^{5/2})$ are the third and fourth components of $U_1^{(5)}$. These are given by

$$U_1^{(5)} = [\,\cdot\,, \cdot\,, -(\mathrm{i}/k)\times\text{third component of } \tilde{U}_1^{(5)}, 0]^{\mathrm{T}}.$$

At $O(\varepsilon^3)$ the non-secular condition (3.21) gives

$$\hat{\mu}_2 \frac{\partial}{\partial \tau_2}|\varphi|^2 + \frac{\partial \mu_4}{\partial \tau_1} - V_g \frac{\partial \mu_4}{\partial \xi_1} + \frac{\partial \lambda_4}{\partial \xi_1} + l_1 \frac{\partial}{\partial \xi_1}|\varphi|^4 = 0$$
$$\hat{\lambda}_2 \frac{\partial}{\partial \tau_2}|\varphi|^2 + \frac{\partial \lambda_4}{\partial \tau_1} - V_g \frac{\partial \lambda_4}{\partial \xi_1} + \frac{\partial \mu_4}{\partial \xi_1} + l_2 \frac{\partial}{\partial \xi_1}|\varphi|^4 + l_3 \frac{\partial}{\partial \xi_1}\left(\varphi \frac{\partial \varphi^*}{\partial \xi_1} - \varphi^* \frac{\partial \varphi}{\partial \xi_1}\right) = 0 \tag{5.8}$$

where $\hat{\lambda}_2$ and $\hat{\mu}_2$ are given by (4.3), and l_1, l_2, l_3 are given in appendix 3. As for λ_2 and μ_2, we assume that λ_4 and μ_4 are independent of τ_1. By combining (5.7) with its complex conjugate it is easily shown that

$$\mathrm{i}\frac{\partial}{\partial \tau_2}|\varphi|^2 + p\frac{\partial}{\partial \xi_1}\left(\varphi^* \frac{\partial \varphi}{\partial \xi_1} - \varphi \frac{\partial \varphi^*}{\partial \xi_1}\right) = \frac{\mathrm{i}}{2}(2m_3 + m_4)\frac{\partial}{\partial \xi_1}|\varphi|^4. \tag{5.9}$$

Now elimination of the τ_2 derivative between (5.8) and (5.9) and an integration with respect to ξ_1 gives a pair of algebraic equations for λ_4, μ_4 from which we find that

$$\frac{\omega^3 \lambda_4}{2} + k\mu_4 = n_2|\varphi|^4 + \mathrm{i}n_3 \frac{\partial}{\partial \xi_1}|\varphi|^2 + \mathrm{i}n_4 \varphi^* \frac{\partial \varphi}{\partial \xi_1} + \nu_4 \tag{5.10}$$

where n_2, n_3, n_4 are given in appendix 3, and ν_4 is an arbitrary real function of τ_2 and ξ_2 that we set to zero just as we set ρ_2 and σ_2 to zero in (4.3). Substitution of (5.10) into (5.7) now gives (1.3) with $q_i = m_i + n_i$ $(i = 2, 3, 4)$. As we are assuming that k is near k_c, p, q_2, q_3, q_4 may be approximated by their values at k_c. These values, together with those for k_c and q_1, are given in table 1. For comparison the table also shows the corresponding results for a Stokes wave taken from Kakutani and Michihiro (1983).

Table 1. Numerical results to four significant figures for ion acoustic waves and Stokes waves. The latter are taken from Kakutani and Michihiro (1983). Note that it is usual to take $\kappa = 0$ for a Stokes wave.

	Ion acoustic wave	Stokes wave
k_c	1.471	1.363
p	−0.1240	−0.1564
q_1	$0.3205\Delta k/\varepsilon$	$5.669\Delta k/\varepsilon$
q_2	163.9	77.40
q_3	−0.4883	1.598
q_4	2.205	3.841
R	$1020\|\varphi_0\|^2 - 6.879\kappa$	$25.87\|\varphi_0\|^2 - 0.6775\kappa$

6. The modulational instability criterion

Using the values of p, q_1, q_2, q_3 and q_4 from table 1, we find that the modulational instability criterion (1.4) may be rearranged to give

$$k > k_c - \varepsilon R \tag{6.1}$$

where k_c and R are also given in the table. The condition (6.1) is the required revision to (4.6), so that the revised critical wavenumber is $k_c - \varepsilon R$. With $\varepsilon^{1/2}|\varphi_0| = 0.015$ and

$\kappa = 0$, for example, $\varepsilon R = 0.2295$ and (6.1) becomes $k > 1.241$. In general, and as discussed in I, if $R > 0$ then wavenumbers with $k_c - \varepsilon R < k < k_c$ that are stable according to (4.6) are in fact unstable, while if $R < 0$ then wavenumbers with $k_c < k < k_c - \varepsilon R$ that are unstable according to (4.6) are in fact stable. However, the coefficient of $|\varphi_0|^2$ in R is alarmingly large so that (6.1) is meaningful only for very small values of $\varepsilon^{1/2}|\varphi_0|$. Nevertheless the result is asymptotically correct. For example, we can certainly conclude that if $\kappa = 0$, so that $R > 0$, then the effect of the non-linearity is to reduce the critical wavenumber. Kakutani and Michihiro (1983)† came to the same conclusion for a Stokes wave, as is indicated in table 1.

7. Concluding remarks

We have investigated a particular example of the general system (1.5), namely a relatively simple system governing ion acoustic plasma wave propagation. We have shown that, while away from marginal modulational instability the modulations of these waves are governed by a NS equation, the modulations near marginal modulational instability are governed by a MNS equation. We have shown how to calculate the correction, due to higher-order non-linearities, to the critical wavenumber for marginal modulational instability. As remarked in I, we expect that the corresponding calculation for any other example of (1.5) will be just as formidable.

It has to be admitted that the results in §§ 5 and 6 are of limited physical significance for actual ion acoustic wave propagation because the model represented by equations (2.1) is over-simplified. Several more sophisticated models have been proposed. Chan and Seshadri (1975) argued that for k near $k_c = 1.471$ (i.e. the value calculated in § 4) electron inertia cannot be neglected. They showed that if it is included in the two-fluid model then k_c is infinite (implying that ion acoustic waves are modulationally stable), but that if the ions have non-zero temperature T_i then k_c is finite and depends on the ratio T_e/T_i. Ichikawa and Taniuti (1973) also allowed for non-zero T_i, but used a Vlasov description of the plasma so that the effects of non-linear Landau damping could be taken into account. Using the reductive perturbation technique they showed that the equation governing the evolution of φ is a NS equation of the form (1.1) but with an additional non-local non-linear integral term due to resonant ions at the group velocity. The effect of these ions also contributes to q. Ichikawa and Watanabe (1977) applied a linear stability analysis to this modified form of the NS equation and showed that the frequency of the modulations ($\tilde{\omega}$ in the notation of I) is complex regardless of the sign of pq, and hence that ion acoustic waves are modulationally unstable when non-linear Landau damping is important.

In view of the above observations we suspect that in a real physical system the evolution of the modulations near the k_c predicted by a two-fluid model with finite T_e/T_i will be determined as a competition between the higher-order non-linearities and wave–particle interaction effects.

Finally we note that Inoue and Matsumoto (1974) have shown that, under certain restrictions, the modulations of wavelike solutions to the general system (1.5) away from marginal modulational instability are governed by a NS equation. Using a

† Johnson (1977) also investigated Stokes waves but found that $R = -25.1|\varphi_0|^2$ when $\kappa = 0$, i.e. almost exactly the negative of Kakutani and Michihiro's result. Consequently he came to a different conclusion regarding the critical wavenumber for marginal modulational instability of a Stokes wave.

generalisation of the methodology given in § 3, we can show that the modulations near marginal modulational instability are governed by a MNS equation. We hope to report this work in due course.

Acknowledgments

I wish to thank Dr Brian Duffy and a referee for helpful comments and suggestions and Mr Roy Veitch for calculating the numerical values for the ion acoustic wave in table 1.

Appendix 1

In (3.6) and (3.12) L_0 is given by

$$L_0 = (-\omega A + kB_0)\frac{\partial}{\partial\vartheta} + \nabla C_0.$$

The $L_m (m=1,2)$ in (3.6) are given by

$$L_1 = kB_1\frac{\partial}{\partial\vartheta} + A\frac{\partial}{\partial\tau_1} + (B_0 - V_g A)\frac{\partial}{\partial\xi_1}$$

$$L_2 = kB_2\frac{\partial}{\partial\vartheta} + B_1\frac{\partial}{\partial\xi_1} + A\frac{\partial}{\partial\tau_2} + (B_0 - V_g A)\frac{\partial}{\partial\xi_2}.$$

The L_m $(m = 1, \ldots, 5)$ in (3.12) are given by

$$L_1 = kB_1\frac{\partial}{\partial\vartheta} \qquad L_2 = kB_2\frac{\partial}{\partial\vartheta} + A\frac{\partial}{\partial\tau_1} + (B_0 - V_g A)\frac{\partial}{\partial\xi_1} \qquad L_3 = kB_3\frac{\partial}{\partial\vartheta} + B_1\frac{\partial}{\partial\xi_1}$$

$$L_4 = kB_4\frac{\partial}{\partial\vartheta} + B_2\frac{\partial}{\partial\xi_1} + A\frac{\partial}{\partial\tau_2} + (B_0 - V_g A)\frac{\partial}{\partial\xi_2} \qquad L_5 = kB_5\frac{\partial}{\partial\vartheta} + B_3\frac{\partial}{\partial\xi_1} + B_1\frac{\partial}{\partial\xi_2}.$$

Appendix 2

The vector coefficients in §§ 4 and 5 that are not given explicitly there are as follows:

$$\alpha^{(2)} = \beta^{(3)} = \begin{bmatrix} (k^2-1)/k^2 \\ -\omega/k^3 \\ -1/k^2 \\ 0 \end{bmatrix} \qquad \beta^{(2)} = \begin{bmatrix} -(12k^4+15k^2+2)/6\omega^2k^2 \\ -(6k^4+9k^2+2)/6\omega k^3 \\ -(3k^4+9k^2+2)/6k^4 \\ \mathrm{i}(3k^4+6k^2+2)/3k^3 \end{bmatrix}$$

$$\alpha^{(3)} = \begin{bmatrix} 2k/\omega^3 \\ 1/\omega^2 \\ 0 \\ 0 \end{bmatrix} \qquad \bar{\gamma}^{(3)} = \mathrm{i}\begin{bmatrix} (24k^4+33k^2+6)/6\omega^4k \\ (6k^4+9k^2+2)/\omega^3k^2 \\ (3k^4+3k^2+2)/6k^5 \\ 0 \end{bmatrix}$$

$$\gamma^{(3)} = \frac{-\mathrm{i}}{(k^4+3k^2+3)}\begin{bmatrix} -(k^6-k^4-7k^2-6)/2\omega^4k \\ (6k^8+21k^6+31k^4+21k^2+6)/6\omega^3k^2 \\ (3k^8+18k^6+34k^4+27k^2+6)/6k^5 \\ 0 \end{bmatrix}$$

$$\delta^{(3)} = -\begin{bmatrix} \mathrm{i}(216k^8+537k^6+405k^4+86k^2+4)/48\omega^2k^5 \\ \mathrm{i}(72k^8+201k^6+181k^4+54k^2+4)/48\omega k^6 \\ \mathrm{i}(24k^8+105k^6+149k^4+54k^2+4)/48k^7 \\ (24k^8+81k^6+93k^4+38k^2+4)/16k^6 \end{bmatrix}$$

$$\alpha^{(4)} = \begin{bmatrix} \mathrm{i}(12k^6-17k^2-4)/3k^5 \\ \mathrm{i}\omega(6k^6-9k^4-24k^2-8)/6k^6 \\ -\mathrm{i}(9k^2+4)/3k^5 \\ (k^4-2k^2-2)/k^4 \end{bmatrix}$$

$$\beta^{(4)} = \frac{1}{(k^4+3k^2+3)} \times \begin{bmatrix} (96k^{16}+12k^{14}-495k^{12}+670k^{10}+4428k^8+5997k^6+3284k^4+688k^2+40)/144\omega^2k^8 \\ (240k^{16}+1218k^{14}+2799k^{12}+4126k^{10}+4626k^8+3903k^6+2096k^4+568k^2+40)/144\omega k^9 \\ (24k^{16}+21k^{14}-9k^{12}+646k^{10}+2265k^8+3081k^6+1940k^4+568k^2+40)/144k^{10} \\ -\mathrm{i}(24k^{16}+21k^{14}-126k^{12}+46k^{10}+1113k^8+2040k^6+1550k^4+508k^2+40)/72k^9 \end{bmatrix}$$

$$\tilde{\alpha}^{(5)} = \mathrm{i}\begin{bmatrix} -1/\omega \\ -k/\omega^2 \\ 0 \\ 0 \end{bmatrix} \qquad \tilde{\beta}^{(5)} = \begin{bmatrix} \mathrm{i}\omega^2/k^3 \\ -\mathrm{i}\omega \\ -\mathrm{i}/k \\ -1 \end{bmatrix} \qquad \tilde{\gamma}^{(5)} = \begin{bmatrix} -\omega^4/k^6 \\ 2\omega^3/k^3 \\ 1/k^2 \\ 0 \end{bmatrix}$$

$$\tilde{\delta}^{(5)} = \frac{\mathrm{i}}{6(k^4+3k^2+3)} \begin{bmatrix} (6k^{10}+15k^8+4k^6-26k^4-27k^2-6)/k^5 \\ (6k^{10}+6k^8-28k^6-76k^4-60k^2-12)/\omega k^4 \\ (3k^8+12k^6+20k^4+15k^2+6)/k^5 \\ 0 \end{bmatrix}$$

$$\tilde{\varepsilon}^{(5)} = \frac{\mathrm{i}}{6(k^4+3k^2+3)} \begin{bmatrix} (6k^8+43k^6+97k^4+81k^2+18)/k^5 \\ -(24k^{10}+57k^8-53k^6-287k^4-246k^2-36)/\omega k^4 \\ (3k^8+12k^6+8k^4-15k^2-18)/k^5 \\ 0 \end{bmatrix}$$

$$\tilde{\zeta}^{(5)} = \frac{1}{288(k^4+3k^2+3)^2} \begin{bmatrix} (240k^{20}+2874k^{18}+14\,817k^{16}+42\,757k^{14}+74\,797k^{12}+79\,523k^{10} \\ +47\,263k^8+10\,489k^6-3200k^4-1920k^2-168)/\omega^2k^8 \\ (912k^{20}+12\,246k^{18}+65\,322k^{16}+188\,054k^{14}+322\,090k^{12}+332\,778k^{10} \\ +195\,026k^8+50\,744k^6-2332k^4-3192k^2-336)/\omega^3k^7 \\ -(24k^{16}+489k^{14}+1845k^{12}+2590k^{10}+469k^8-2417k^6 \\ -2480k^4-848k^2-56)(k^4+3k^2+3)/k^{10} \\ 0 \end{bmatrix}$$

Appendix 3

The coefficients m_2, m_3, m_4 in (5.7) are given by

$$\{576(k^4+3k^2+3)^2k^{12}\}m_2 = \omega^3\{1152k^{24}+17\,424k^{22}+111\,507k^{20}+405\,648k^{18}+935\,264k^{16} + 1427\,294k^{14}+1450\,004k^{12}+951\,352k^{10}+367\,547k^8+60\,596k^6 - 6220k^4-3408k^2-336\}$$

$$\{12(k^4+3k^2+3)k^7\}m_3=\omega^2\{12k^{12}+33k^{10}+0k^8-114k^6-169k^4-90k^2-12\}$$

$$\{12(k^4+3k^2+3)k^7\}m_4=-\omega^3\{24k^{12}+75k^{10}-48k^8-492k^6-719k^4-366k^2-36\}.$$

The coefficients l_1, l_2, l_3 in (5.8) are given by

$$\{36(k^4+3k^2+3)^2k^6\}l_1 = \{36k^{16}+369k^{14}+1641k^{12}+4119k^{10}+6301k^8 + 5883k^6+3129k^4+774k^2+36\}$$

$$\{18(k^4+3k^2+3)^2\omega^3k^5\}l_2 = \{36k^{16}+414k^{14}+1875k^{12}+4506k^{10}+6259k^8 + 5046k^6+2265k^4+576k^2+108\}$$

$$\{\omega k^2\}l_3=\mathrm{i}\{k^2-2\}.$$

The coefficients n_2, n_3, n_4 in (5.10) are given by

$$\{72(k^4+3k^2+3)^3\omega^3k^6\}n_2 = \{144k^{22}+2124k^{20}+13\,467k^{18}+48\,825k^{16}+111\,904k^{14}+167\,905k^{12} + 163\,957k^{10}+99\,442k^8+33\,852k^6+5820k^4+1080k^2+288\}$$

$$\{2(k^4+3k^2+3)^2\omega k^3\}n_3=\{2k^6+6k^4+7k^2+2\}\{k^6+k^4-6k^2-12\} \qquad n_4=-2n_3.$$

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